On differential formulas of second degree that admit an integration*

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§1 Among such differential formulas of second degree that admit an integration this formula is especially remarkable:

$$\frac{(x\partial x+y\partial y)(\partial y\partial\partial x-\partial x\partial\partial y)}{(\partial x^2+\partial y^2)^{\frac{3}{2}}},$$

which, if x and y denote orthogonal coordinates of a curve, arises, if the element $x\partial x+y\partial y$ is divided by the radius of curvature of this curve; since it is known that the integral of this form is

$$\frac{y\partial x-x\partial y}{\sqrt{\partial x^2+\partial y^2}},$$

as will become clear to anyone doing the calculation, if the differential of this formula is calculated. Therefore, since the integration is not obvious by any means, and indeed requires many detours, I decided to discuss this topic more accurately here, whence one will be able to understand, how many other formulas of this kind be found which likewise admit an integration.

§2 That this can be done more easily, let us eliminate the differentials of second degree from the calculation, which is most conveniently achieved by putting $\partial y=p\partial x$ such that instead of the second differentials this new

*Original Title: "De formulis differentialibus secundi gradus quae integrationem admittunt", first published in: *Nova Acta Academiae Scientiarum Imperialis Petropolitanae, Volume 11* (1798, written 1773): pp. 3– 26, reprint in: Opera Omnia: Series 1, Volume 23, pp. 313 – 338, Eneström Number E700, translated by: Alexander Aycock for the "Euler-Kreis Mainz".

quantity $p = \frac{\partial y}{\partial x}$ is introduced into the calculation, which contains the ratio of the first differentials, of course. Therefore, then it will be

$$x\partial x + y\partial y = \partial x(x + py)$$

and

$$\partial x^2 + \partial y^2 = \partial x^2(1 + pp),$$

and hence the denominator of the given formula becomes

$$(\partial x^2 + \partial y^2)^{\frac{3}{2}} = \partial x^3(1 + pp)^{\frac{3}{2}};$$

finally, for the other factor of the numerator one has

$$\partial y\partial\partial x = p\partial x\partial\partial x,$$

and because of

$$\partial\partial y = p\partial\partial x + \partial p\partial x$$

it will be

$$\partial x\partial\partial y = p\partial x\partial\partial x + \partial p\partial x^2,$$

and so that other factor will be

$$\partial y\partial\partial x - \partial x\partial\partial y = -\partial p\partial x^2,$$

having substituted which the given formula will take this form:

$$-\frac{\partial p(x + py)}{(1 + pp)^{\frac{3}{2}}},$$

whose integral will therefore be

$$\frac{y\partial x - x\partial y}{\sqrt{\partial x^2 + \partial y^2}} = \frac{y - px}{\sqrt{1 + pp}},$$

whose differential yields the above formula, of course.

§3 Therefore, since after this substitution only the one differential ∂p enters into the difference-differential formula, I willconsider this formula: $V\partial p$ in general and will investigate values of what kind have to be attributed to this letter V that the integral of the formula $V\partial p$ can be exhibited; here it is certainly evident that this quantity V has to be a certain function of the three variables x, y and p, of which nature which hence have to be for the integration to succeed I therefore decided to investigate here more accurately.

§4 And first from that what I once taught about the integrability of differential formulas of higher orders one can without any difficulty exhibit criteria from which one can decide whether such a formula $V\partial p$ admits an integration or not. But at that time I considered such a form $\int Z\partial x$, where having put

$$\partial y = p\partial x, \quad \partial p = q\partial x, \quad \partial q = r\partial x, \quad \partial r = s\partial x \quad \text{etc.}$$

the letter Z denotes a function constructed arbitrarily from the letters x, y, p, q, r, s etc., and I showed that, as often of this formula $\int Z\partial x$ was integrable, then it will always be

$$0 = \left(\frac{\partial Z}{\partial y}\right) - \frac{1}{\partial x}\partial \cdot \left(\frac{\partial Z}{\partial p}\right) + \frac{1}{\partial x^2}\partial\partial \cdot \left(\frac{\partial Z}{\partial q}\right) - \frac{1}{\partial x^3}\partial^3 \cdot \left(\frac{\partial Z}{\partial r}\right) + \frac{1}{\partial x^4}\partial^4 \cdot \left(\frac{\partial Z}{\partial s}\right) - \text{etc.},$$

But if that quantity does not become equal to zero directly, then that equation expresses the relation between x and y, for which the integral formula $\int Z\partial x$ has a maximum or minimum value.

§5 Therefore, to reduce the formula $\int V\partial p$ that we consider here to this form: $\int Z\partial x$, let us set $\partial p = q\partial x$ such that our formula becomes $Vq\partial x$ and hence $Z = Vq$; let us remark here that the quantity V contains only the three letters x, y and p; having observed this it will be

$$\left(\frac{\partial Z}{\partial y}\right) = \left(\frac{q\partial V}{\partial y}\right);$$

furthermore,

$$\left(\frac{\partial Z}{\partial p}\right) = \left(\frac{q\partial V}{\partial p}\right) \quad \text{and} \quad \left(\frac{\partial Z}{\partial q}\right) = V,$$

and so the criterion indicating integrability will be

$$0 = \left(\frac{q\partial V}{\partial y}\right) - \frac{1}{\partial x}\partial \cdot \left(\frac{q\partial V}{\partial p}\right) + \frac{1}{\partial x^2}\partial\partial \cdot V,$$

which equation can also be represented this way:

$$0 = q\left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x}\partial \cdot \left[q\left(\frac{\partial V}{\partial p}\right) - \frac{1}{\partial x}\partial V\right],$$

but then, because of $q\partial x = \partial p$, it can also be represented as follows:

$$0 = \partial p\left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x}\partial \cdot \left[\partial p\left(\frac{\partial V}{\partial p}\right) - \partial V\right].$$

§6 Therefore, since in general by means of the now sufficiently established symbols

$$\partial V = \partial x\left(\frac{\partial V}{\partial x}\right) + \partial y\left(\frac{\partial V}{\partial y}\right) + \partial p\left(\frac{\partial V}{\partial p}\right),$$

having substituted this value the desired criterion will be expressed this way:

$$0 = \partial p\left(\frac{\partial V}{\partial y}\right) + \frac{1}{\partial x}\partial \cdot \left[\partial x\left(\frac{\partial V}{\partial x}\right) + \partial y\left(\frac{\partial V}{\partial y}\right)\right],$$

which equation therefore contains the desired criterion such that, as often as that formula actually becomes equal to zero, we can always be certain that the given formula $V\partial p$ is integrable.

§7 Since V, by assumption, is a function involving these three variables x, y and p, differentiating in usual manner we have

$$\partial V = M\partial x + N\partial y + P\partial p$$

and the criterion will be contained in this equation:

$$0 = N\partial p + \partial \cdot (M + Np),$$

which is further expanded into this one:

$$0 = 2N\partial p + \partial M + p\partial N.$$

To see the power of this more clearly, let us apply that criterion to the formula given initially

$$\frac{\partial p(x+py)}{(1+pp)^{\frac{3}{2}}},$$

where, since

$$V = \frac{x+py}{(1+pp)^{\frac{3}{2}}},$$

having taken just x as variable, one finds

$$M = \frac{1}{(1+pp)^{\frac{3}{2}}},$$

but having taken just y as variable, it will be

$$N = \frac{p}{(1+pp)^{\frac{3}{2}}};$$

therefore, hence it will be

$$\partial M = -\frac{3p\partial p}{(1+pp)^{\frac{5}{2}}} \quad \text{and} \quad \partial N = \frac{(1-2pp)\partial p}{(1+pp)^{\frac{3}{2}}},$$

having substituted which values, since

$$\begin{aligned} &1. \quad 2N\partial p = \frac{2p\partial p}{(1+pp)^{\frac{3}{2}}} = \frac{2p\partial p(1+pp)}{(1+pp)^{\frac{3}{2}}}, \\ &2. \quad \partial M = -\frac{3p\partial p}{(1+pp)^{\frac{5}{2}}}, \\ &3. \quad p\partial N = \frac{p\partial p(1-2pp)}{(1+pp)^{\frac{5}{2}}}, \end{aligned}$$

the sum of these formulas is obviously zero. From this it is understood that this formula is indeed integrable, even though its integral is not known.

§8 But since our goal here rather is to find appropriate values to be taken for V, for which the differential formula $V\partial p$ admits an integration, the found criterion has no use; for this reason, let us begin our investigation from the simplest cases, in which the given formula admits an integration, among which without a doubt the simplest one is that one, in which V denotes a constant quantity. Therefore, let $V = 1$, and it will be $\int \partial p = p$. But hence it further follows, if the differential ∂p is multiplied by an arbitrary function of this integral p, which we will call $\Delta : p$, then this formula $\partial p \Delta : p$ will be integrable, which is certainly clear per se. For, here the under term *integrability* we do not just understand what can be exhibited algebraically, but in general what can be assigned in terms of arbitrary transcendental quantities.

§9 But second most simple case, in which the formula $V \partial p$ becomes integrable, is the case $V = x$ such that the differential formula is $= x \partial p$. For, since by the all-known reduction $\int x \partial p = px - \int p \partial x$, because of $p \partial x = \partial y$ this integral will be $\int x \partial p = px - y$. Therefore, if hence $\Delta : (x - y)$ denotes an arbitrary function of the formula $px - y$, also this differential formula: $x \partial p \Delta : (px - y)$ extending a lot further will admit an integration, which having put $px - y = V$ because of $\partial V = x \partial p$ takes this form: $\partial V \Delta : V$.

§10 Furthermore, even a third very simple case is given in which our formula $V \partial p$ becomes integrable, which arises by putting $V = \frac{y}{pp}$. For, by the same reduction, by which $\int t \partial u = tu - \int u \partial t$, taking $t = y$ and $\partial u = \frac{\partial p}{pp}$, whence $\partial t = \partial y = p \partial x$ and $u = \frac{-1}{p}$, it will be

$$\int \frac{y \partial p}{pp} = \frac{-y}{p} + \int \partial x = x - \frac{y}{p}.$$

Therefore, if further $\Delta : \left(x - \frac{y}{p}\right)$ denotes an arbitrary function of the formula $y - \frac{y}{p}$, even this a lot more general differential formula will be integrable:

$$\frac{y \partial p}{pp} \Delta : \left(x - \frac{y}{p}\right).$$

For, if one sets $x - \frac{y}{p} = V$, because of $\partial V = \frac{y \partial p}{pp}$, this form becomes $= \partial V \Delta : V$, which is obviously always integrable.

§11 Having constituted these principal cases, let us also investigate more complicated cases, in which the general formula $V \partial p$ will likewise become integrable; for this purpose, let us go through the following problems.

PROBLEM 1

Let two functions of p be in questions, which we will call P and Q, of such a nature that this differential formula: $\partial p(Px+Qy)$ becomes integrable.

SOLUTION

§12 Since this formula involves two parts, let us expand them separately by the mentioned reduction, and first it will certainly be

$$\int Px\partial p = x\int P\partial p - \int \partial x \int P\partial p,$$

where the integral $\int P\partial p$ can be considered as a known quantity, since P denotes a function of p. In like manner, for the other part it will be

$$\int Qy\partial p = y\int Q\partial p - \int \partial y \int Q\partial p,$$

where the last terms contain formulas that are not integrable per se on both sides, whence it is necessary that after having collected these two formulas into one sum these two last terms cancel each other. Therefore, let

$$\int \partial x \int P\partial p + \int \partial y \int Q\partial p = 0,$$

and hence by integrating, because if $\partial y = p\partial x$, it will be

$$\int P\partial p + p\int Q\partial p = 0.$$

Now let us differentiate again and we will obtain

$$P + \int Q\partial p + Qp = 0,$$

which differentiated again yields

$$\partial P + p\partial Q + 2Q\partial p = 0,$$

which equation contains the relation between the two functions P and Q in question.

§13 Therefore, if this last equation is multiplied by p,

$$p\partial P + \partial \cdot Qpp = 0$$

will result; hence it is clear, if the one of these two functions P and Q were known, from this the other can be determined. For, if for the sake of illustration, the function P was given, because of $\int p\partial P + Qpp = C$ it will be

$$Q = \frac{C - \int p\partial P}{pp}.$$

But if the other function Q was given, from the first formula it will be

$$\partial P = -p\partial Q - 2Q\partial p,$$

and hence by integrating

$$P = C - \int (p\partial Q + 2Q\partial p)$$

or even

$$P = C - Qp - \int Q\partial p.$$

§14 But whenever those two function P and Q had been determined correctly that way, then the integral of the given differential formula $\partial p(Px + Qy)$ will be expressed in this way that $= x\int P\partial p + y\int Q\partial p$. And we already noted that the one of the functions P and Q can be assumed arbitrarily. Yes, one can even be constitute a certain relation between P and Q. If, for the sake of an example, we want that $P = nQp$, having substituted these values in this differential equation, it will be

$$(n+2)Q\partial p + (n+1)p\partial Q = 0,$$

whence one further deduces

$$\frac{(n+2)\partial p}{p} + \frac{(n+1)\partial Q}{Q} = 0,$$

whose integral is

$$(n+2)\log p + (n+1)\log Q = \log C,$$

and hence further $p^{n+2}Q^{n+1} = C$, from which one deduces

$$Q = \frac{C}{p^{\frac{n+2}{n+1}}}, \quad \text{consequently} \quad P = \frac{nC}{p^{\frac{1}{n+1}}}.$$

§15 Since the integral was found to be $x\int P\partial p + y\int Q\partial p$, these two integral formulas are to be considered to obtain two constants such that the true integral results expressed this way: $x\int P\partial p + y\int Q\partial p + \alpha x + \beta y$, where the constants α and β have to be determined for each case in such a way that, after having taken the differentials, the element ∂x exits the calculation, what happens, if it was

$$\partial x \int P\partial p + p\partial x \int Q\partial p + \alpha\partial x + \beta p\partial x = 0,$$

whence, as we already found

$$P\partial p + \partial p \int Q\partial p + Qp\partial p + \beta\partial p = 0,$$

which divided by ∂p and differentiated again yields

$$\partial P + 2Q\partial p + p\partial Q = 0,$$

which equation expressed the required relation between P and Q.

Another Solution of the same Problem

§16 Since $x\partial p$ is the differential of the formula $px - y$, by reduction it will be

$$\int Pxd\partial = P(px-y) - \int (px-y)\partial P;$$

further, since $\frac{y\partial p}{pp}$ is the differential of the formula $px - y$, by reduction it will be:

$$\int Qy\partial y = \int Qpp \cdot \frac{y\partial p}{pp} = Qpp\left(x - \frac{y}{p}\right) - \int \left(x - \frac{y}{p}\right)\partial \cdot Qpp.$$

Therefore, combining them the integral of the given formula will be

$$P(px-y) + Qpp\left(x - \frac{y}{p}\right) - \int (px-y)\partial P - \int \left(x - \frac{y}{p}\right)\partial \cdot Qpp,$$

where it is evident that the last integral parts must be equal to zero. Hence, having taken the differentials, one has to set

$$(px-y)\partial P+\left(x-\frac{y}{p}\right)\partial\cdot Qpp=0,$$

which equation divided by $px-y$ gives

$$\partial P+\frac{1}{p}\partial\cdot Qpp=0$$

or

$$\partial P+p\partial Q+2Q\partial p=0,$$

which is the same equation between P and Q that the first solution produced.

§17 Since we saw above that this formula

$$\frac{(x+py)\partial p}{(1+pp)^{\frac{3}{2}}}$$

admits an integration, after an application here it will be

$$P=\frac{1}{(1+pp)^{\frac{3}{2}}}\quad\text{and}\quad Q=\frac{p}{(1+pp)^{\frac{3}{2}}}.$$

Now let us consider the quantity P as known and let us see, whether we find the same value for Q. Therefore, since

$$\partial P=\frac{-3p\partial p}{(1+pp)^{\frac{5}{2}}},$$

the found equation will become

$$\frac{-3p\partial p}{(1+pp)^{\frac{5}{2}}}+p\partial Q+2Q\partial p=0,$$

which multiplied by p yields

$$\partial\cdot Qpp=\frac{3pp\partial p}{(1+pp)^{\frac{5}{2}}}\quad\text{and hence}\quad Qpp=\int\frac{3pp\partial p}{(1+pp)^{\frac{5}{2}}}.$$

But paying little attention, it will be clear that

$$\int \frac{3pp\partial p}{(1+pp)^{\frac{5}{2}}} = \frac{p^3}{(1+pp)^{\frac{3}{2}}},$$

and so it will be

$$Qpp = \frac{p^3}{(1+pp)^{\frac{3}{2}}}$$

and hence

$$Q = \frac{p}{(1+pp)^{\frac{3}{2}}} + \frac{C}{pp}.$$

§18 Therefore, hence we see that for the value

$$P = \frac{1}{(1+pp)^{\frac{3}{2}}}$$

not only

$$Q = \frac{p}{(1+pp)^{\frac{3}{2}}},$$

but, in more generality, one can take

$$Q = \frac{p}{(1+pp)^{\frac{3}{2}}} + \frac{C}{pp}$$

such that this formula admits an integration. Therefore, since the integral was found in general to be

$$P(px-y) + Qpp\left(x - \frac{y}{p}\right),$$

having substituted these values the integral will be

$$\frac{px-y}{(1+pp)^{\frac{3}{2}}} + \frac{pp(px-y)}{(1+pp)^{\frac{3}{2}}} + \frac{C(px-y)}{p},$$

which is reduced to this form:

$$\frac{px-y}{\sqrt{1+pp}} + \frac{C(px-y)}{p}.$$

Problem 2

If M and N were any arbitrary given functions of p, to find a function Π of the same letter such that this differential formula: $(Mx + Ny)\Pi\partial p$ admits an integration.

Solution

§19 If we compare this problem with the preceding one, it is immediately clear that the functions denoted by the letters P and Q are $M\Pi$ and $N\Pi$ such that $P = M\Pi$ and $Q = N\Pi$. Therefore, since integrability requires this equation:

$$\partial P + 2Q\partial p + p\partial Q = 0,$$

after this substitution we will obtain the following equation:

$$M\partial\Pi + \Pi\partial M + 2N\Pi\partial p + Np\partial\Pi + \Pi p\partial N = 0,$$

from which, since M and N are known functions of p, we find

$$\frac{\partial\Pi}{\Pi} = \frac{-\partial M - 2N\partial p - p\partial N}{M + Np},$$

whence by integrating we calculate

$$\log\Pi = -\log(M + Np) - \int \frac{N\partial p}{M + Np}.$$

Therefore, for the sake of brevity, let us put

$$\int \frac{N\partial p}{M + Np} = \log K,$$

since even this formula K can be considered as given, and so it will be $\log\Pi = -\log(M + Np) - \log K + \log A$. Therefore, for the solution of our problem we will have:

$$\Pi = \frac{A}{K(M + Np)}, \quad \text{while} \quad \log K = \int \frac{N\partial p}{M + Np}.$$

§20 But having found this value of the function in question

$$\Pi = \frac{A}{K(M+Np)},$$

since above the integral in general resulted as

$$P(px-y) + Qpp\left(x - \frac{y}{p}\right) = (px-y)(P+Qp),$$

having substituted the respective values for P and Q, the integral of the given differential formula $(Mx+Ny)\Pi\partial p$ will be

$$(px-y)(M\Pi + N\Pi p) = \frac{A(px-y)(M+Np)}{K(M+Np)},$$

which is conveniently further reduced to this very simple form:

$$\frac{A(px-y)}{K},$$

and so it will be

$$\int \frac{(Mx+Ny)\partial p}{K(M+Np)} = \frac{px-y}{K},$$

while

$$\log K = \int \frac{N\partial p}{M+Np} \quad \text{or} \quad K = e^{\int \frac{N\partial p}{M+Np}},$$

which will be worth the effort to illustrate it with examples.

Example 1

§21 Let $M=1$ and $N=1$ such that this differential formula is propounded: $(x+y)\Pi\partial p$. Therefore, it will be

$$\log K = \int \frac{\partial p}{1+p} = \log(1+p)$$

here and hence $K = 1+p$ such that now the function in question is $\Pi = \frac{A}{(1+p)^2}$, and hence the differential formula admitting an integration will be $\frac{(x+y)\partial p}{(1+p)^2}$, whose integral obviously is $\frac{px-y}{1+p}$. For, if this formula is differentiated,

$$\frac{x\partial p}{1+p} - \frac{(px-y)\partial p}{(1+p)^2},$$

which is reduced to this form:

$$\frac{(x+y)\partial p}{(1+p)^2}.$$

EXAMPLE 2

§22 Let both functions M and N be constants, namely $M = m$ and $N = n$, such that this differential formula is given: $(mx+ny)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{n\partial p}{m+np} = \log(m+np)$$

first such that $K = m + np$. Hence the function Π in question will be

$$\frac{A}{(m+np)^2}$$

such that this formula is already integrable:

$$\frac{(mx+ny)\partial p}{(m+np)^2},$$

whose integral will be

$$\frac{px-y}{m+np},$$

of course.

EXAMPLE 3

§23 Now let us take $M = 1$ and $N = p$ that this formula is to be rendered integrable $(x+py)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{p\partial p}{1+pp} = \log\sqrt{1+pp}$$

first and hence $K = \sqrt{1+pp}$, whence the function in question becomes

$$\Pi = \frac{A}{(1+pp)^{\frac{3}{2}}},$$

and thus the differential formula admitting an integration will be

$$\frac{(x+py)\partial p}{(1+pp)^{\frac{3}{2}}}, \tag{1}$$

which is the one we considered initially, whose integral therefore is

$$\frac{px-y}{\sqrt{1+p^2}}.$$

EXAMPLE 4

§24 Now let $M = n$ and $N = np$ such that the formula that is to rendered integrable is $(mx + npy)\Pi\partial y$. Therefore, here it will be

$$\log K = \int \frac{np\partial p}{m+npp} = \log\sqrt{m+npp}$$

and hence $K = \sqrt{m+npp}$, whence the function in question will be

$$\Pi = \frac{A}{(m+npp)^{\frac{3}{2}}}$$

such that this formula

$$\frac{(mx+npy)\partial p}{(m+npp)^{\frac{3}{2}}}$$

is already integrable, whose integral will therefore be

$$\frac{px-y}{\sqrt{m+npp}}.$$

EXAMPLE 5

§25 Now let $M = m$ and $N = np^{\lambda-1}$ such that formula that is to be rendered integrable is $(mx + np^{\lambda-1}y)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{np^{\lambda-1}\partial p}{m+np^{\lambda}} = \frac{1}{\lambda}\log(m+np^{\lambda})$$

and hence $K = (m + np^\lambda)^{\frac{1}{\lambda}}$, whence the function Π in question will be

$$\frac{A}{(m + np^\lambda)^{\frac{\lambda+1}{\lambda}}}$$

such that this formula is already integrable:

$$\frac{(mx + np^{\lambda-1}y)\partial p}{(m + np^\lambda)^{\frac{\lambda+1}{\lambda}}},$$

whose integral will therefore be

$$\frac{px - y}{(m + np^\lambda)^{\frac{1}{\lambda}}}.$$

EXAMPLE 6

§26 Now let $M = mp$ and $N = n$ such that the formula that is to rendered integrable is $(mpx + ny)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{n\partial p}{mp + np} = \frac{n}{m+n} \log p$$

and hence $K = p^{\frac{n}{m+n}}$; therefore,

$$\Pi = \frac{A}{(m+n)p^{\frac{m+2n}{m+n}}},$$

and so this formula is integrable now

$$\frac{(mpx + ny)\partial p}{(m+n)p^{\frac{m+2n}{m+n}}},$$

whose integral will therefore be

$$\frac{px - y}{p^{\frac{n}{m+n}}}.$$

§27 Here the especially remarkable case occurs, in which $m = -n$ or $m + n = 0$; for, then because of the infinite exponent of p a highly incongruent formula results. But this case is obvious per se. For, if one finds Π that this formula $(px - y)\Pi\partial p$ become integrable, since $\partial \cdot (px - y) = x\partial p$, it is evident that there is no function of just p, which can satisfy this condition. But as soon as it was not $m + n = 0$, the solution is always possible.

EXAMPLE 7

§28 Now take $M = mpp$ and $N = n$ such that this formula has to be rendered integrable: $(mppx + ny)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{n\partial p}{np + mpp} = \log p - \log(mp + n),$$

consequently

$$K = \frac{p}{mp + n}, \quad \text{and hence} \quad \Pi = \frac{A}{pp},$$

and so the integrable formula will be

$$\frac{(mppx + ny)\partial p}{pp};$$

for, its integral will be

$$\frac{(px - y)(mp + n)}{p}.$$

EXAMPLE 8

§29 Now let $M = p^{\lambda+1}$ and $N = 1$ such that the formula $(p^{\lambda+1}x + y)\Pi\partial p$ is to be rendered integrable. Therefore, here it will be

$$\log K = \int \frac{\partial p}{p^{\lambda+1} + p} = \log p - \frac{1}{\lambda}\log(p^{\lambda} + 1),$$

thus,

$$K = \frac{p}{(p^{\lambda+1} + 1)^{\frac{1}{\lambda}}},$$

and hence

$$\Pi = \frac{A(p^{\lambda} + 1)^{\frac{1-\lambda}{\lambda}}}{pp},$$

whence the integrable formula will be

$$\frac{(p^{\lambda} + 1)^{\frac{1-\lambda}{\lambda}}(p^{\lambda+1}x + y)\partial p}{pp},$$

whose integral obviously is

$$\frac{(px-y)(p^\lambda+1)^{\frac{1}{\lambda}}}{p}.$$

EXAMPLE 9

§30 Finally, let $M = mp^{\lambda+1}$ and $N = n$ such that the formula to be rendered integrable is $(mp^{\lambda+1}x + ny)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{n\partial p}{mp^{\lambda+1}+np} = \log p - \frac{1}{\lambda}\log(mp^\lambda + n),$$

and hence

$$K = \frac{p}{(mp^\lambda+n)^{\frac{1}{\lambda}}},$$

and thus

$$\Pi = \frac{A(mp^\lambda+n)^{\frac{1-\lambda}{\lambda}}}{pp},$$

whence the integrable formula will be

$$\frac{(mp^{\lambda+1}x+ny)(mp^\lambda+n)^{\frac{1-\lambda}{\lambda}}\partial p}{pp},$$

whose integral will be

$$\frac{(px-y)(mp^\lambda+n)^{\frac{1}{\lambda}}}{p},$$

of course.

PROBLEM

To find two functions of p, which we will call P and Q, such that this differential formula: $(px-y)^{n-1}(Px+Qy)\partial p$ becomes integrable.

SOLUTION

§31 Since $x\partial p = \partial \cdot (px - y)$, it will be

$$\int Px\partial p(px-y)^{n-1} = \frac{1}{n}P(px-y)^n - \frac{1}{n}\int (px-y)^n \partial P.$$

Further, since

$$\frac{y\partial p}{pp} = \partial \cdot \left(x - \frac{y}{p}\right),$$

let us write $Qpp \cdot \frac{y\partial p}{pp}$ instead of $Qy\partial p$, but then let us write $p\left(x - \frac{y}{p}\right)$ instead of $px - y$, and hence one will have to write $p^{n-1}\left(x - \frac{p}{y}\right)^{n-1}$ instead of $(px - y)^{n-1}$. Therefore, hence for the one part we will have

$$Qy\partial p(px-y)^{n-1} = Qpp \cdot \frac{y\partial p}{pp} \cdot p^{n-1}\left(x - \frac{y}{p}\right)^{n-1} = Qp^{n+1} \cdot \frac{y\partial p}{pp} \cdot \left(x - \frac{y}{p}\right)^{n-1},$$

and hence by reduction it will be

$$\int Qy\partial p(px-y)^{n-1} = \frac{1}{n}Qp^{n+1}\left(x - \frac{y}{p}\right)^n - \frac{1}{n}\int \left(x - \frac{y}{p}\right)^n \partial \cdot Qp^{n+1}.$$

§32 Therefore, for the given formula to admit an integration it now is necessary that the two last summatory terms become zero, whence this equation arises:

$$(px-y)^n \partial P + \left(x - \frac{y}{p}\right)^n \partial \cdot Qp^{n+1} = 0,$$

and hence dividing by $(px-y)^n$ it will be

$$p^n \partial P + \partial \cdot Qp^{n+1} = 0,$$

whose expansion yields

$$\partial P + p\partial Q + (n+1)Q\partial p = 0,$$

which equation contains the required relation between P and Q; therefore, hence given the one the other can be determined at the same time; for; then the integral of the propounded formula will be

$$\frac{1}{n}P(px-y)^n + \frac{1}{n}Qp^{n+1}\left(x-\frac{y}{p}\right)^n$$

or

$$\frac{1}{n}(px-y)^n(P+Qp).$$

PROBLEM 4

If M and N denote arbitrary given functions of p, to find a function Π such that this differential formula: $(px-y)^{n-1}(Mx+Ny)\Pi\partial p$ becomes integrable.

SOLUTION

§33 The solution of the preceding problem is transferred to this one by setting $P = M\Pi$ and $Q = N\Pi$, whence the condition found before will lead to this equation:

$$M\partial\Pi + \Pi\partial M + Np\partial\Pi + \Pi p\partial N + (n+1)N\Pi\partial p = 0,$$

from which one finds

$$\frac{\partial\Pi}{\Pi} = \frac{-\partial M - p\partial N - (n+1)N\partial p}{M+Np},$$

which integrated yields

$$\log\Pi = -\log(M+Np) - n\int\frac{N\partial p}{M+Np}.$$

§34 Now let us, as we did above, put

$$\int\frac{N\partial p}{M+Np} = \log K,$$

and going back to numbers, it will be

$$\Pi = \frac{A}{K^n(M+Np)},$$

and so our integrable formula will be

$$\frac{(px-y)^{n-1}(Mx+Ny)\partial p}{K^n(M+Np)}.$$

For, its integral will be

$$\frac{1}{n}\frac{(px-y)^n(M+Np)}{K^n(M+Np)} = \frac{(px-y)^n}{nK^n},$$

whence for $n = 1$ obviously the case of the third problem emerges.

§35 Here the especially remarkable case occurs, in which $n = 0$; for, then because of $K^n = 1$ the integral formula that was rendered integrable will be

$$\frac{(Mx+Ny)\partial p}{(M+Np)(px-y)}.$$

But its integral hence seems to become infinite, values of which kind are reduced to logarithms; for, the formula $\frac{(px-y)^0}{0}$ is equivalent to $\log(px-y)$. Nevertheless this integral is not satisfactory by any means, the reason for which lies hidden in the vanishing of the number n; but one finds this differential formual to be resolved into

$$\frac{x\partial p}{px-y} - \frac{N\partial p}{M+Np};$$

hence if, as we did before, we put

$$\int \frac{N\partial p}{M+Np} = \log K,$$

its integral will be $\log(px-y) - \log K$ such that in this case the integral is $\log \frac{px-y}{K}$. But in the remaining cases the integrals will be algebraic, for which reason we will consider the following examples.

EXAMPLE 1

§36 Let $M = 1$ and $N = 1$, and as before it will be $\log K = \int \frac{\partial p}{1+p} = \log(1+p)$, and hence $K = 1+p$, and hence $\Pi = \frac{A}{(1+p)^{n+1}}$, whence our now integrable formula will be

$$\frac{(px-y)^{n-1}(x+y)\partial p}{(1+p)^{n+1}},$$

whose integral is

$$\frac{(px-y)^n}{n(1+p)^n}.$$

EXAMPLE 2

§37 Now let us put $M = \alpha$ and $N = \beta$ such that the formula to be rendered integrable is $(px-y)^{n-1}(\alpha x + \beta y)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{\beta \partial p}{\alpha + \beta p} = \log(\alpha + \beta p)$$

and hence $K = \alpha + \beta p$; and thus $\Pi = \frac{A}{(\alpha+\beta p)^{n+1}}$, whence our formula that is to be rendered integrable will be

$$\frac{(px-y)^{n-1}(\alpha x + \beta y)\partial p}{(\alpha + \beta p)^{n+1}},$$

whose integral is

$$\frac{(px-y)^n}{n(\alpha+\beta p)^n}.$$

EXAMPLE 3

§38 Now let $M = 1$ and $N = p$ such that the formula to be rendered integrable is $(px-y)^{n-1}(x+py)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{p\partial p}{1+pp} = \log \sqrt{1+pp}$$

and hence $K = \sqrt{1+pp}$ and thus

$$\Pi = \frac{A}{(1+pp)^{\frac{n+2}{2}}},$$

and so our integrable formula will be

$$\frac{(px-y)^{n-1}(x+py)\partial p}{(1+pp)^{\frac{n+2}{2}}};$$

for, its integral will be

$$\frac{(px-y)^n}{n(1+pp)^{\frac{n}{2}}}.$$

EXAMPLE 4

§39 Now let $M = \alpha$ and $N = \beta p$ such that the formula that is to be rendered integrable is $(px-y)^{n-1}(\alpha x + \beta py)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{\beta p \partial p}{\alpha + \beta pp} = \frac{1}{2}\log(\alpha + \beta pp)$$

and hence $K = \sqrt{\alpha + \beta pp}$, whence the function Π in question will be

$$\frac{A}{(\alpha+\beta pp)^{\frac{n+2}{2}}}.$$

Hence our integrable formula will be

$$\frac{(px-y)^{n-1}(\alpha x+\beta py)\partial p}{(\alpha+\beta pp)^{\frac{n+2}{2}}},$$

whose integral obviously will be

$$\frac{(px-y)^n}{n(\alpha+\beta pp)^{\frac{n}{2}}}.$$

EXAMPLE 5

§40 Let $M = \alpha$ and $N = \beta p^{\lambda-1}$ such that formula that is to be rendered integrable is $(px-y)^{n-1}(\alpha x + \beta p^{\lambda-1}y)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{\beta p^{\lambda-1}\partial p}{\alpha + \beta p^{\lambda}} = \frac{1}{\lambda}\log(\alpha + \beta p^{\lambda})$$

and hence $K = (\alpha + \beta p^{\lambda})^{\frac{1}{\lambda}}$, whence the function Π in question will be

$$\frac{A}{(\alpha + \beta p^{\lambda})^{\frac{n+\lambda}{\lambda}}},$$

and so our integrable formula will be

$$\frac{(px - y)^{n-1}(\alpha x + \beta p^{\lambda-1} y)\partial p}{(\alpha + \beta p^{\lambda})^{\frac{n+\lambda}{\lambda}}},$$

whose integral obviously is

$$\frac{(px - y)^{n}}{n(\alpha + \beta p^{\lambda})^{\frac{n}{\lambda}}}.$$

EXAMPLE 6

§41 Now let $M = \alpha p$ and $N = \beta$ such that the formula to be rendered integrable is $(px - y)^{n-1}(\alpha p x + \beta y)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{\beta \partial p}{\alpha p + \beta p} = \frac{\beta}{\alpha + \beta} \log p$$

and hence $K = p^{\frac{\beta}{\alpha+\beta}}$. Therefore, hence the propounded function Π will be

$$\Pi = \frac{A}{(\alpha + \beta) p^{\frac{\alpha + (n+1)\beta}{\alpha+\beta}}},$$

and so our integrable formula will now be

$$\frac{(px - y)^{n-1}(\alpha p x + \beta y)\partial p}{(\alpha + \beta) p^{\frac{\alpha + (n+1)\beta}{\alpha+\beta}}},$$

whose integral therefore is

$$\frac{(px - y)^{n}}{n p^{\frac{\beta n}{\alpha+\beta}}}.$$

EXAMPLE 7

§42 Now take $M = \alpha pp$ and $N = \beta$ such that this formula: $(px-y)^{n-1}(\alpha ppx + \beta y)\Pi\partial p$ has to be rendered integrable. Therefore, here it will be

$$\log K = \int \frac{\beta\partial p}{\alpha pp + \beta p} = \log p - \log(\alpha p + \beta),$$

as a logical consequence $K = \frac{p}{\alpha p+\beta}$, and hence

$$\Pi = \frac{A(\alpha p + \beta)^{n-1}}{p^{n+1}},$$

and so the integrable formula will now be

$$\frac{(px-y)^{n-1}(\alpha ppx + \beta y)(\alpha p + \beta)^{n-1}\partial p}{p^{n+1}},$$

whose integral obviously is

$$\frac{(px-y)^n(\alpha p + \beta)^n}{np^n}.$$

EXAMPLE 8

§43 Now let $M = p^{\lambda+1}$ and $N = 1$ such that the formula that is to be rendered integrable is $(px-y)^{n-1}(p^{\lambda+1}x + y)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{\partial p}{p^{\lambda+1} + p} = \log p - \frac{1}{\lambda}\log(p^\lambda + 1),$$

consequently $K = \frac{p}{(p^\lambda+1)^{\frac{1}{\lambda}}}$, and hence

$$\Pi = \frac{A(p^\lambda + 1)^{\frac{n-\lambda}{\lambda}}}{p^{n+1}},$$

whence the integrable formula will be

$$\frac{(px-y)^{n-1}(p^{\lambda+1}x + y)(p^\lambda + 1)^{\frac{n-\lambda}{\lambda}}\partial p}{p^{n+1}},$$

whose integral will hence be

$$\frac{(px-y)^n(p^\lambda + 1)^{\frac{n}{\lambda}}}{np^n}.$$

EXAMPLE 9

§44 Finally, let $M = \alpha p^{\lambda+1}$ and $N = \beta$ such that formula that is to be rendered integrable is $(px - y)^{n-1}(\alpha p^{\lambda+1}x + \beta y)\Pi\partial p$. Therefore, here it will be

$$\log K = \int \frac{\beta \partial p}{\alpha p^{\lambda+1} + \beta p} = \log p - \frac{1}{\lambda}\log(\alpha p^{\lambda} + \beta)$$

and hence $K = \frac{p}{(\alpha p^{\lambda}+\beta)^{\frac{1}{\lambda}}}$ and thus,

$$\Pi = \frac{A(\alpha p^{\lambda} + \beta)^{\frac{n-\lambda}{\lambda}}}{p^{n+1}},$$

whence the integrable formula will be

$$\frac{(p - xy)^{n-1}(\alpha p^{\lambda+1}x + \beta y)(\alpha p^{\lambda} + \beta)^{\frac{n-\lambda}{\lambda}}\partial p}{p^{n+1}},$$

whose integral will therefore be

$$\frac{(px - y)^{n}(\alpha p^{\lambda} + \beta)^{\frac{n}{\lambda}}}{np^{n}}.$$